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Fundamental frequency of a standing heavy plate with vertical simply-supported edges

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Abstract

The vibration of a rectangular plate under self-weight (or acceleration) is studied. The vertical edges are simply supported and a semi-analytic Levy-integration method is used. The fundamental frequencies are determined for various top and bottom boundary conditions. It is found that self-weight has considerable effect on both the frequency and the mode shape.

1. Introduction

The study of the vibration of a standing plate under its own weight is important for the design of walls, panels, and windows of buildings. Also, accelerating mobile structures could generate body forces equivalent to gravity. There exist some reports on the buckling of standing plates [1–4] but there seems to be only one reference on the vibration of standing plates. Herrmann [5] used an energy method and considered the vibration of a standing rectangular plate simply supported on all sides, but only one term was used for the frequency results.

The present paper studies the vibration of a standing plate simply supported on the two vertical sides. Four kinds of practical boundary conditions will be considered. The bottom edge, bearing the total weight, is clamped or simply supported, and the top edge, bearing no load, is either free or simply supported.

Since the two vertical edges are simply supported, the Levy method would reduce the plate equations to a linear ordinary differential equation with non-constant coefficients. We shall use an integration method similar to that of Barasch and Chen [6] which is easier than the Ritz method.

2. Formulation

Normalize all lengths by the plate height L. Fig. 1(a) shows the standing plate with Cartesian axes at the lower left corner. Due to gravity, the body force per length in the vertical direction is $-\rho g L(1-y)$ where ρ is

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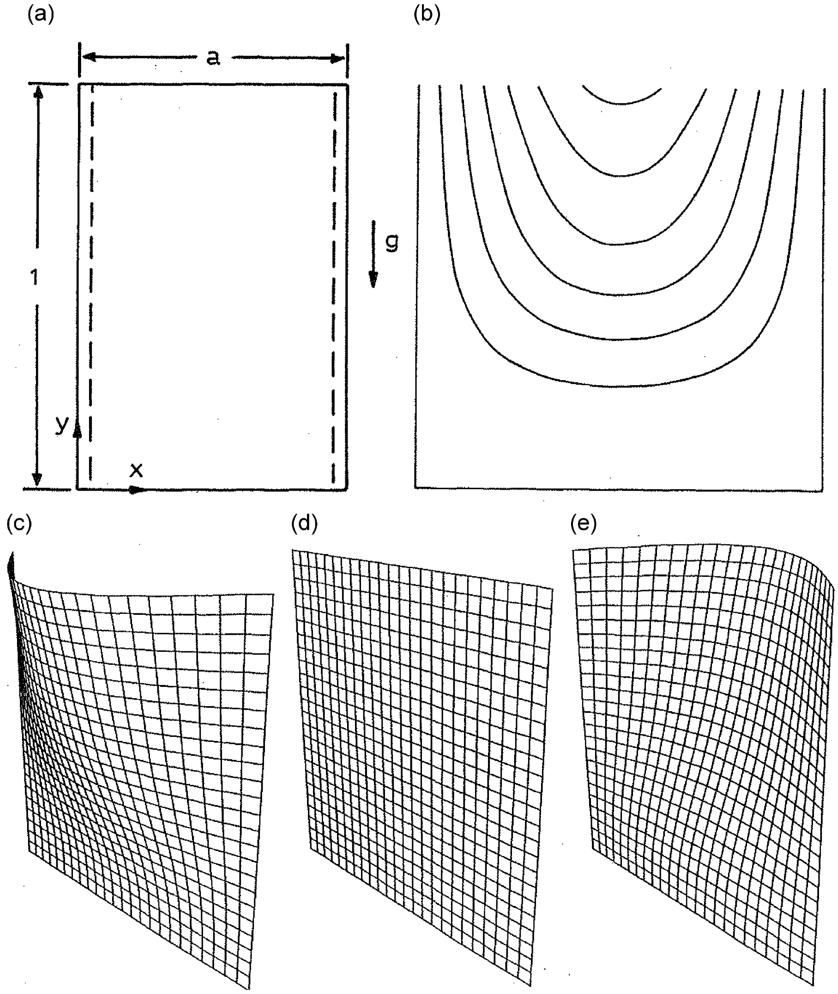

Fig. 1. (a) The standing plate with coordinate axes and normalized dimensions. (b) Relative amplitude for the square C–F (clamped bottom-free top) plate. $\gamma = 100$, $k = 1.3877$. (c–e) 3D modes at $\Omega t = \pi/2, 0, -\pi/2$, respectively.

the mass per area and g is the gravitational acceleration. Then the governing equation is [7]

$$\nabla^4 w + \gamma \frac{\partial}{\partial y}\left[(1-y)\frac{\partial w}{\partial y}\right] - k^4 w = 0 \tag{1}$$

Here w is the lateral deflection, the weight parameter γ and the frequency parameter k are defined as

$$\gamma = \frac{\rho g L^3}{D}, \quad k^4 = \frac{\rho \Omega^2 L^4}{D} \tag{2}$$

where D is the flexural rigidity and Ω is the frequency. Since the vertical sides are simply supported, set

$$w(x, y) = \sin(\alpha x) Y(y) \tag{3}$$

where $\alpha = n\pi/a$. Eq. (1) becomes

$$Y'''' - 2\alpha^2 Y'' + \alpha^4 Y + \gamma[(1-y)Y']' - k^4 Y = 0 \tag{4}$$

For given α, γ and the bottom, top boundary conditions, the lowest eigenvalue k, or the square root of normalized frequency, is sought.

If the bottom is clamped and the top is free, the boundary conditions are

$$Y(0)=0, \quad Y'(0)=0 \tag{5}$$

$$Y''(1)-\alpha^2 v Y(1)=0, \quad Y'''(1)-\alpha^2(2-v)Y'(1)=0 \tag{6}$$

Here Eq. (6) represents zero moment and zero shear and v is the Poisson ratio. The two point boundary value problem is very difficult to solve, even numerically. We shall turn it into two initial value problems. Let Y_1 satisfy Eq. (4) and the initial conditions

$$Y_1(0)=0, \quad Y_1'(0)=0, \quad Y_1''(0)=1, \quad Y_1'''(0)=0 \tag{7}$$

and Y_2 satisfy Eq. (4) and the initial conditions

$$Y_2(0)=0, \quad Y_2'(0)=0, \quad Y_2''(0)=0, \quad Y_2'''(0)=1 \tag{8}$$

Then the solution to Eqs. (4) and (5) can be expressed as a linear combination of the two independent solutions Y_1 and Y_2

$$Y=C_1Y_1+C_2Y_2 \tag{9}$$

Substitution of Eq. (9) into Eq. (6) gives two linear homogeneous algebraic equations

$$\begin{pmatrix} Y_1''(1)-\alpha^2 vY_1(1) & Y_2''(1)-\alpha^2 vY_2(1) \\ Y_1'''(1)-\alpha^2(2-v)Y_1'(1) & Y_2'''(1)-\alpha^2(2-v)Y_2'(1) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=0 \tag{10}$$

For non-trivial C_1 and C_2, the determinant of coefficients is set to zero, which yields a nonlinear equation in k. Using bisection, the lowest k (giving the fundamental frequency) is thus obtained. Since the amplitude of the eigenfunctions is indeterminate, we can set $C_1=1$ and solve for C_2 from one of the equations in Eq. (10). Then Eqs. (3) and (9) give the mode shapes.

In all cases we find $n=1$ gives the fundamental frequency. We used $v=0.3$ for Poisson ratio and a standard Runge–Kutta algorithm for the solutions Y_1 and Y_2. The relative error is set at 10^{-6}.

For the case where the bottom is clamped and the top is simply supported, Eq. (6) is replaced by

$$Y(1)=0, \quad Y''(1)=0 \tag{11}$$

If the bottom is simply supported, Eqs. (7) and (8) are replaced by

$$Y_1(0)=0, \quad Y_1'(0)=1, \quad Y_1''(0)=0, \quad Y_1'''(0)=0 \tag{12}$$

$$Y_1(0)=0, \quad Y_1'(0)=0, \quad Y_1''(0)=0, \quad Y_1'''(0)=1 \tag{13}$$

The eigenvalue is obtained from the top condition, Eq. (6) if it is free or Eq. (11) if it is simply supported. The mode shapes are found similarly.

3. Results and discussions

Let the first letter denote the bottom boundary condition and the second letter denote the top boundary condition. Thus C–S means bottom clamped and top simply supported and S–F means the bottom is simply supported and the top is free.

When the frequency is zero ($k=0$) our method gives the buckling load. Table 1 shows the results for C–F, S–F, C–S and S–S cases. Notice that our results for C–F and S–F cases are in complete agreement with those of Ref. [4] who used a Ritz method. On the other hand, the one term solution of Herrmann [5] for the S–S case and the empirical formulas of Kato [9] for the column are too inaccurate to be useful.

Typical mode shapes for the vibration of a square C–F plate are shown in Fig. 1(b–e). For narrower and heavier plates, the largest amplitude may occur near the bottom, as shown in Fig. 2 for the S–S plate. If weight were absent, the vibration of an S–S plate would be symmetric about the mid point. Multiple cells in the vertical direction also occur for heavier and narrower plates.

The frequency parameters k for C–F plate are given in Table 2. Here $\gamma=0$ indicates the self-weight is absent, and the frequencies have been reported by many authors [7]. In the Appendix A we briefly present the

Table 1
The buckling load for various widths a.

a	C–F	S–F	C–S	S–S
0.2	1414.6*	1162.5*	1414.6	1162.5
0.5	306.09*	213.72*	307.83	214.02
1	102.22*	62.701*	111.96	63.783 (26.32)
5	9.4528*	1.2010*	54.380	19.995 (16.48)
10	8.2246*	0.28234*	52.966	18.921 (18.67)
1000	7.8374*	0.00003*	52.501	18.569 (19.74)
∞	7.83735$^{\bullet}$ [7.85]	0	52.5006 [60.34]	18.5687 [26.57]

The small dot is from the exact solution [8]. The asterisked values are identical with [4], the values in parenthesis are from [5] and the values in brackets are from [9].

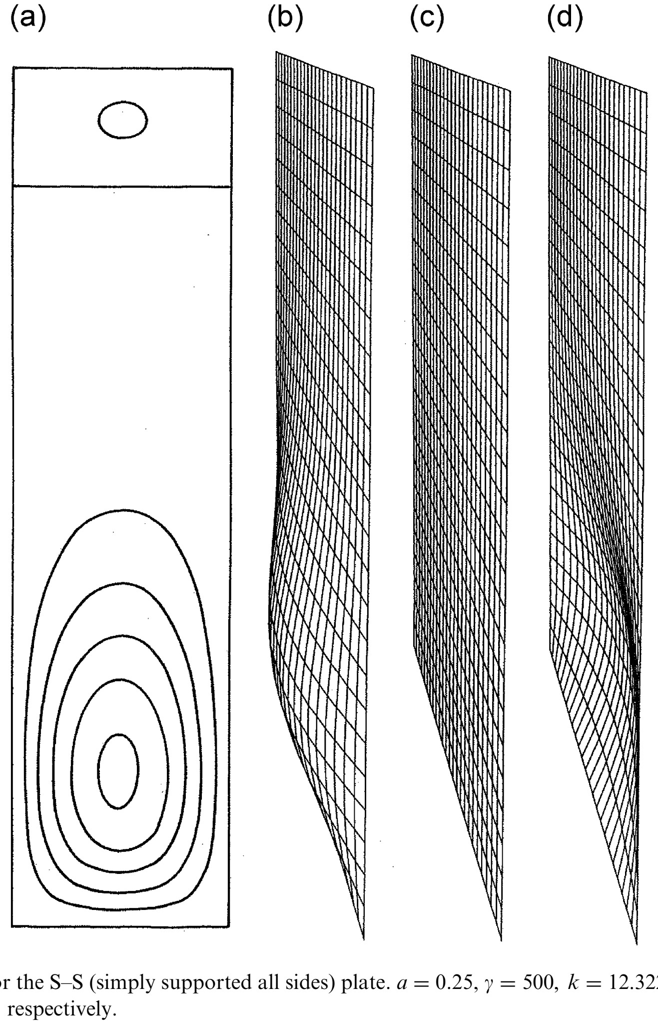

Fig. 2. (a) Relative amplitude for the S–S (simply supported all sides) plate. $a = 0.25$, $\gamma = 500$, $k = 12.322$. Note interior nodal line. (b–d) 3D modes at $\Omega t = \pi/2, 0, -\pi/2$, respectively.

exact solution. When a is small, the frequency becomes independent of self-weight, and approaches infinity as $a \to 0$. When a is large, or very wide width, the frequency becomes independent of the aspect ratio, and decreases to some constant value depending on γ, provided $\gamma < 7.8384$. If $\gamma > 7.8384$ the frequency decreases as width a increases, until k reaches zero, whereby the plate buckles due to self-weight.

Table 2
Frequency parameter k for the C–F case.

a	γ						
	0	7	10	20	100	300	1000
0.2	15.740	15.739	15.739	15.738	15.729	15.710	15.169
0.5	6.458	6.448	6.443	6.429	6.306	3.458	0
1	3.562	3.501	3.474	3.378	1.388	0	0
2	2.388	2.158	2.031	1.092	0	0	0
5	1.964	1.403	0	0	0	0	0
10	1.897	1.180	0	0	0	0	0
20	1.881	1.103	0	0	0	0	0
1000	1.875	1.078	0	0	0	0	0
∞	1.875	1.073	0	0	0	0	0

The zero entry means the plate has already buckled.

Table 3
Frequency parameter k for the S–F case.

a	γ						
	0	1	2	10	50	200	500
0.2	15.735	15.734	15.734	15.734	15.730	15.714	15.681
0.5	6.419	6.417	6.416	6.404	6.341	4.491	0
1	3.418	3.409	3.399	3.316	2.623	0	0
2	2.008	1.960	1.907	0.958	0	0	0
5	1.160	0.742	0	0	0	0	0
10	0.8069	0	0	0	0	0	0
20	0.5682	0	0	0	0	0	0
1000	0.0802	0	0	0	0	0	0
∞	0	0	0	0	0	0	0

The zero entry means the plate has already buckled.

The frequency for the S–F case is given in Table 3. In this case there is no constant asymptote for any γ as $a \to \infty$. The plate always buckles. Again the results for $\gamma = 0$ can be obtained from the Appendix A.

The frequency for the C–S case is given in Table 4. Here the $\gamma = 0$ column is exact, and the plate does not buckle if $\gamma < 52.501$.

The frequency for the S–S case is given in Table 5. The plate would not buckle for large a if $\gamma < 18.569$.

We find the fundamental frequency becomes independent of the top and bottom edge conditions when the plate is very narrow. Our exact solution equation (A.6) for the S–S plate show that the frequency (in all cases) becomes infinite algebraically, i.e.

$$k \sim \pi / a \quad \text{as } a \to 0 \tag{14}$$

For fixed width as the weight (or acceleration) increases, the frequency decreases until $k = 0$, when the plate buckles statically. On the other hand, for fixed weight and increased width, the frequency decreases and reaches zero (buckle) for the S–S case, but the plate buckles only for larger self-weight in the other three cases.

The semi-analytical method used in this paper would also determine the higher frequencies. We note that if the vertical edges are not simply supported, the Ritz method is recommended instead of numerical integration such as finite elements.

Table 4
Frequency parameter k for the C–S case.

a	γ					
	0	10	50	100	300	1000
0.2	16.048	16.045	16.045	16.014	15.956	15.169
0.5	7.189	7.158	7.027	6.839	3.734	0
1	4.863	4.764	4.265	2.901	0	0
2	4.163	4.004	2.933	0	0	0
5	3.964	3.778	2.145	0	0	0
10	3.936	3.745	1.944	0	0	0
20	3.929	3.737	1.884	0	0	0
1000	3.927	3.734	1.863	0	0	0
∞	3.927	3.734	1.863	0	0	0

The zero entry means the plate has already buckled.

Table 5
Frequency parameter k for the S–S case.

a	γ					
	0	18	25	50	200	500
0.2	16.019	16.014	16.012	16.004	15.955	15.825
0.5	7.025	6.959	6.931	6.827	4.523	0
1	4.443	4.152	4.011	3.196	0	0
2	3.512	2.766	2.092	0	0	0
5	3.204	1.831	0	0	0	0
10	3.157	1.508	0	0	0	0
20	3.146	1.386	0	0	0	0
1000	3.142	1.337	0	0	0	0
∞	π	1.337	0	0	0	0

The zero entry means the plate has already buckled.

Appendix A. Exact solutions when $\gamma = 0$

When weight is absent Eq. (4) becomes an ordinary differential equation with constant coefficients

$$Y'''' - 2\alpha^2 Y'' + \alpha^4 Y - k^4 Y = 0 \tag{A.1}$$

The solution is a linear combination of

$$\{\sinh(\sqrt{\alpha^2 + k^2}y), \cosh(\sqrt{\alpha^2 + k^2}y), \sinh(\sqrt{\alpha^2 - k^2}y), \cosh(\sqrt{\alpha^2 - k^2}y)\} \quad \text{if } k < \alpha \tag{A.2}$$

$$\{\sinh(\sqrt{2}\alpha y), \cosh(\sqrt{2}\alpha y), y, 1\} \quad \text{if } k = \alpha \tag{A.3}$$

$$\{\sinh(\sqrt{\alpha^2 + k^2}y), \cosh(\sqrt{\alpha^2 + k^2}y), \sin(\sqrt{k^2 - \alpha^2}y), \cos(\sqrt{k^2 - \alpha^2}y)\} \quad \text{if } k > \alpha \tag{A.4}$$

But for the fundamental frequency, we find $k > \alpha$ is always true. Thus the form of Eq. (A.4) is used. Let $\lambda = \sqrt{\alpha^2 + k^2}$, $\mu = \sqrt{k^2 - \alpha^2}$.

If the bottom is simply supported, the general solution satisfying $Y(0) = 0$, $Y''(0) = 0$ is

$$Y = A \sinh(\lambda y) + B \sin(\mu y) \tag{A.5}$$

For the S–S case, we substitute Eq. (A.5) into Eq. (11) and find $A = 0$ and $\sin\mu = 0$. This yields $\mu = m\pi$ where m is an integer. Thus the frequency parameter is $k = \sqrt{\alpha^2 + m^2\pi^2}$ and the fundamental frequency is

$$k = \pi\sqrt{1 + 1/a^2} \tag{A.6}$$

For the S–F case, Eq. (6) gives the characteristic equation

$$\mu(\lambda^2 - \alpha^2 v)[\mu^2 + \alpha^2(2 - v)]\sinh\lambda\cos\mu - \lambda(\mu^2 + \alpha^2 v)(\lambda^2 - \alpha^2(2 - v)]\cosh\lambda\sin\mu = 0 \tag{A.7}$$

Here the eigenvalue k can be obtained by bisection.

If the bottom is clamped, the general solution satisfying Eq. (5) is

$$Y = A[\mu\sinh(\lambda y) - \lambda\sin(\mu y)] + B[\cosh(\lambda y) - \cos(\mu y)] \tag{A.8}$$

For the C–S case substitute Eq. (A.8) into Eq. (11) and the characteristic equation is

$$(\mu\sinh\lambda - \lambda\sin\mu)(\lambda^2\cosh\lambda + \mu^2\cos\mu) - \mu\lambda(\cosh\lambda - \cos\mu)(\lambda\sinh\lambda + \mu\sin\mu) = 0 \tag{A.9}$$

The C–F case is a bit more involved. Eqs. (A.8), (6) yield

$$\begin{aligned}&[\mu\lambda(\lambda\sinh\lambda + \mu\sin\mu) - \alpha^2 v(\mu\sinh\lambda - \lambda\sin\mu)][\lambda^3\sinh\lambda - \mu^3\sin\mu - \alpha^2(2 - v)(\lambda\sinh\lambda + \mu\sin\mu)]\\ &\quad - [\lambda^2\cosh\lambda + \mu^2\cos\mu - \alpha^2 v(\cosh\lambda - \cos\mu)]\mu\lambda[\lambda^2\cosh\lambda + \mu^2\cos\mu - \alpha^2(2 - v)(\cosh\lambda - \cos\mu)] = 0\end{aligned} \tag{A.10}$$

Eqs. (A.6), (A.7), (A.9) and (A.10) are used in the determination of the frequency parameter when $\gamma = 0$.

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